CALCULUS.

397. Proposed by C. N. SCHMALL, New York City.

On the radii vectores of one loop of the lemniscate $\rho^2 = a^2 \cos 2\theta$ as diameters, circles are described passing through the pole. Find the locus of their points of intersection, and show that the area is twice that of the loop.

Solution by Theodore Howard, New Haven, Conn.

Let ϕ be the angle between the polar axis and the radius vector of the circle described on the radius vector of the lemniscate as a diameter. Let r be the radius vector of the circle, the origin and polar axis being the same as for the lemniscate. Then the equation of the circle is $r = \rho \cos(\phi - \theta)$. Or $r^2 = \rho^2 \cos^2(\phi - \theta) = a^2 \cos 2\theta \cos^2(\phi - \theta)$, substituting from the equation of the lemniscate. In this equation, θ is the variable parameter. Taking the derivative of this equation with respect to θ, we have

$$0 = 2a^2 \cos(\phi - \theta)[\sin(\phi - \theta)\cos 2\theta - \sin 2\theta \cos(\phi - \theta)],$$

$$= 2a^2 \cos(\phi - \theta)\sin(\phi - 3\theta).$$

Hence, $\theta = \dfrac{\phi}{3}$, or $\theta = \phi - \dfrac{\pi}{2}$ an extraneous value. Substituting $\dfrac{\phi}{3}$ for θ in the preceding equation, we have $r^2 = a^2 \cos^3 \dfrac{2\phi}{3}$, the equation of the envelope. The area of one loop of the envelope is $A = \dfrac{a^2}{2}\int_b^a \cos^3 \dfrac{2\phi}{3} d\phi = 3a^2 \int_0^{\pi/4} \cos^3 2\theta d\theta = a^2$. The area of one loop of the lemniscate is

$$A' = a^2 \int_0^{\pi/4} \cos 2\theta d\theta = \frac{a^2}{2}.$$

Hence,

$$A = 2A'.$$

Also solved by C. E. Horne, Paul Capron, H. L. Agard, and Norman Anning.

398. Proposed by V. M. SPUNAR, Chicago, Ill.

Solve

$$2\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = 0.$$

Solution by W. W. Beman, University of Michigan.

Assume $z = \phi(y + mx)$. The equation in m is $2m^2 - 3m - 2 = 0$. Hence,

$$z = \phi(y + 2x) + \psi(2y - x).$$

Also solved by Elijah Swift, J. L. Riley, and the Proposer.

399. Proposed by B. J. BROWN, Victor, Colorado.

A cow is tethered by a perfectly smooth rope, a slip noose in the rope being thrown over a large square post. If the cow pulls the rope taut in the direction shown in the figure, at what angle will the rope leave the post?

From Granville's *Diff. and Int. Calculus*, p. 120, Prob. 55.

Solution by H. L. Agard, Williams College.

The rope leaves the post in such a manner that the knot at the end of the noose is in front of the middle of one face of the post. Let a be the thickness of the post, φ the angle at which the rope leaves the post, b the distance from the knot to the corner of the post, and c the perpendicular distance from the knot to the post.

Then

$$b = \frac{a}{2}\sec\varphi \qquad \text{and} \qquad c = \frac{a}{2}\tan\varphi.$$

If l is the total length of the rope and x is the distance of the cow from the post,

$$x = l - 3a - 2b + c = l - 3a - a \sec \varphi + \frac{a}{2} \tan \varphi. \quad (1)$$

Taking the derivative of x with respect to φ and setting the result equal to 0, we have,

$$\frac{dx}{a\varphi]} = -a \sec \varphi \cdot \tan \varphi + \frac{a}{2} \sec^2 \varphi = 0.$$

Hence, $\sec \varphi = 0$ and $\sec \varphi - 2 \tan \varphi = 0$. Whence, φ from the last equation $= \pi/6$. This value of ϕ in $d^2x/d\varphi^2$ gives a negative value. Hence, for $\varphi = \pi/6$, x is a maximum.

Also solved by J. A. BULLARD.

NOTE. It seems to us, that the statement of this problem is misleading. As stated, it suggests a problem in mechanics and as such it is not a problem of maxima and minima except incidentally.

When the rope is taut, the tension in the three parts of the rope about the knot are equal and the knot (better a smooth ring) is then in stable equilibrium. From this fact, it follows that the rope leaves the post at an angle of 30°, and it no more involves the idea of maxima and minima than does the fact that a cube of homogeneous density has its center of gravity in the lowest position when a face of the cube is in contact with a horizontal plane.

However, it turns out in this problem that if the length of the rope is greater than $\frac{1}{3}(9 + 2\sqrt{3})$ times the length of a side of a right section of the post, the cow will be at a maximum distance from the side of the post, when the rope is taut.

To establish the reason why these two facts should coexist transcends the ability of the average student of elementary calculus.

The problem may be stated in definite form as follows: An inextensible weightless string of length $l > na$, where n is an integer greater than unity, is to be cut into two parts. The ends of one part of the string are fastened to the points A and B located in the same horizontal line, the distance between them being a. To one end of the other part of the string is attached a small weight while the other end of this part of the string is attached to the middle point of the suspended part, and the parts and weight are then allowed to hang freely. What angle will the string make with line AB when it is so cut that the weight is at a maximum distance from the line AB?

It may be argued in defense of the problem as given, that it leaves the student to discover for himself the particular quantity that becomes a maximum, instead of calling his attention to that quantity, as the proposed modification does, which is an argument worthy of consideration. But the statement of this problem leads the student by suggestion away from the discovery of the quantity which is to be made a maximum rather than directs him towards such discovery, which seems to us to be objectionable. EDITOR FINKEL.

MECHANICS.

314. Proposed by C. N. SCHMALL, New York City.

A rectangular box of height h, and having a plane mirror for its bottom, contains a quantity of water of unknown height x. In the lid are two small apertures distant $2a$ from each other. A ray of light entering one aperture with an angle of incidence φ, emerges, after refraction and reflection, through the other aperture. If μ be the index of refraction of water, show that the height of the water is

$$x = \frac{h \tan \varphi - a}{\tan \varphi - \dfrac{\sin \varphi}{(\mu^2 - \sin^2 \varphi)^{1/2}}}.$$

SOLUTION BY FRANK IRWIN, University of California.

We are given, in the figure, $AF = a$, $CF = h$, $CE = x$. Then $BD = h - x$. $a = AD + BE = (h - x) \tan \varphi + x \tan \theta$. Whence $x = (h \tan \varphi - a)/(\tan \varphi - \tan \theta)$.

Now $\mu = \sin \varphi/\sin \theta$, or $\mu \sin \theta = \sin \varphi$; so that

$$\tan \theta = \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}} = \frac{\sin \varphi}{\sqrt{\mu^2 - \sin^2 \varphi}}.$$